

# Math 23: Differential Equations

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Placement Test

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This is my preparation for the placement test at Dartmouth College before Matriculation so that I can enroll in [Math 54: Topology](#), and the prerequisite is [Math 23: Differential Equations](#). I am following the syllabus from [Winter 2024](#) to self-study.

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# 1 Introduction

## 1.1 Some Basic Mathematical Models: Direction Fields

### Definition 1.1.1 (Direction Field)

A **direction field** (slope field) for  $y' = f(t, y)$  is a plot where at each point  $(t, y)$  we draw a short line segment with slope  $f(t, y)$ .

**Remark 1.1.1.** • **Equilibrium solutions:** horizontal lines where  $f(t, y) = 0$ .

- **Isoclines:** curves where  $f(t, y) = c$  (constant slope).
- Solution curves follow the field and never cross (by uniqueness).

## 1.2 Solutions to Some Differential Equations

### Definition 1.2.1 (Solution)

$y = \phi(t)$  is a **solution** of  $y' = f(t, y)$  on  $I$  if  $\phi'(t) = f(t, \phi(t))$  for all  $t \in I$ . A solution satisfying  $y(t_0) = y_0$  solves the **initial value problem** (IVP).

### Definition 1.2.2 (General vs. Particular Solution)

The **general solution** contains arbitrary constants (the full family). A **particular solution** pins down those constants via initial conditions.

## 1.3 Classification of Differential Equations

**Definition 1.3.1 (Classification)** 1. **Type:** **ODE** = one independent variable; **PDE** = several.

2. **Order:** highest derivative present.  $y'' + y = 0$  is 2nd order.

3. **Linearity:** linear if  $a_n(t)y^{(n)} + \dots + a_0(t)y = g(t)$  with  $a_i, g$  depending only on  $t$ . Otherwise **nonlinear** (e.g.  $yy' = 1$ ).

## 2 First Order Differential Equations

### 2.1 Linear Equations: Method of Integrating Factors

#### Definition 2.1.1 (First-order Linear Differential Equation)

A **first-order linear differential equation** is an equation that can be written in the standard form:

$$y' + p(t)y = g(t)$$

where  $p(t)$  and  $g(t)$  are given functions of the independent variable  $t$ . The equation is called **linear** because the dependent variable  $y$  and its derivative  $y'$  appear to the first power and are not multiplied together.

If  $g(t) = 0$ , the equation is called **homogeneous**:

$$y' + p(t)y = 0$$

Otherwise, it is called **non-homogeneous**.

#### Fact 2.1.1 (Method of Integrating Factors)

To solve the first-order linear differential equation  $y' + p(t)y = g(t)$ :

1. Calculate the **integrating factor**:  $\mu(t) = e^{\int p(t)dt}$
2. Multiply both sides of the equation by  $\mu(t)$ :

$$\mu(t)y' + \mu(t)p(t)y = \mu(t)g(t)$$

3. Recognize that the left side is the derivative of  $\mu(t)y$ :

$$\frac{d}{dt}[\mu(t)y] = \mu(t)g(t)$$

4. Integrate both sides:

$$\mu(t)y = \int \mu(t)g(t)dt + C$$

5. Solve for  $y$ :

$$y = \frac{1}{\mu(t)} \left[ \int \mu(t)g(t)dt + C \right]$$

#### Example 2.1.2 (Standard Form Example)

Solve the differential equation  $y' + 2ty = t^2$ .

*Solution.* This is already in standard form with  $p(t) = 2t$  and  $g(t) = t^2$ .

**Step 1:** Find the integrating factor:

$$\mu(t) = e^{\int 2t dt} = e^{t^2}$$

**Step 2:** Multiply the equation by  $\mu(t)$ :

$$e^{t^2} y' + 2te^{t^2} y = t^2 e^{t^2}$$

**Step 3:** Recognize the left side as a derivative:

$$\frac{d}{dt}[e^{t^2} y] = t^2 e^{t^2}$$

**Step 4:** Integrate both sides:

$$e^{t^2} y = \int t^2 e^{t^2} dt$$

Using integration by parts or substitution, we get:

$$e^{t^2} y = \frac{1}{2} e^{t^2} (t^2 - 1) + C$$

**Step 5:** Solve for  $y$ :

$$y = \frac{1}{2}(t^2 - 1) + C e^{-t^2}$$

□

**Example 2.1.3 (Equation Not in Standard Form)**

Solve the differential equation  $(4 + t^2) \frac{dy}{dt} + 2ty = 4t$ .

*Solution.* First, we need to put this in standard form by dividing by  $(4 + t^2)$ :

$$\frac{dy}{dt} + \frac{2t}{4 + t^2} y = \frac{4t}{4 + t^2}$$

Now we have  $p(t) = \frac{2t}{4+t^2}$  and  $g(t) = \frac{4t}{4+t^2}$ .

**Step 1:** Find the integrating factor:

$$\mu(t) = e^{\int \frac{2t}{4+t^2} dt}$$

Let  $u = 4 + t^2$ , then  $du = 2t dt$ :

$$\int \frac{2t}{4 + t^2} dt = \int \frac{1}{u} du = \ln |u| = \ln(4 + t^2)$$

Therefore:  $\mu(t) = e^{\ln(4+t^2)} = 4 + t^2$

**Step 2:** Multiply by the integrating factor:

$$(4 + t^2) \frac{dy}{dt} + 2ty = 4t$$

**Step 3:** The left side is  $\frac{d}{dt}[(4 + t^2)y]$ :

$$\frac{d}{dt}[(4 + t^2)y] = 4t$$

**Step 4:** Integrate:

$$(4 + t^2)y = \int 4t dt = 2t^2 + C$$

**Step 5:** Solve for  $y$ :

$$y = \frac{2t^2 + C}{4 + t^2}$$

□

**Example 2.1.4 (Initial Value Problem)**

Find the general solution of the differential equation

$$\frac{dy}{dt} - 2y = 4 - t$$

and solve the initial value problem with  $y(0) = 1$ . Discuss the behavior as  $t \rightarrow \infty$ .

*Solution.* This is in standard form with  $p(t) = -2$  and  $g(t) = 4 - t$ .

**Step 1:** Find the integrating factor:

$$\mu(t) = e^{\int -2dt} = e^{-2t}$$

**Step 2-3:** Multiply and recognize:

$$\frac{d}{dt}[e^{-2t}y] = e^{-2t}(4 - t)$$

**Step 4:** Integrate the right side using integration by parts:

$$\begin{aligned} \int e^{-2t}(4 - t)dt &= \int 4e^{-2t}dt - \int te^{-2t}dt \\ &= -2e^{-2t} - \left(-\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t}\right) \\ &= -2e^{-2t} + \frac{1}{2}te^{-2t} + \frac{1}{4}e^{-2t} \\ &= e^{-2t} \left(-2 + \frac{t}{2} + \frac{1}{4}\right) = e^{-2t} \left(\frac{t}{2} - \frac{7}{4}\right) \end{aligned}$$

Therefore:

$$e^{-2t}y = e^{-2t} \left(\frac{t}{2} - \frac{7}{4}\right) + C$$

**Step 5:** General solution:

$$y = \frac{t}{2} - \frac{7}{4} + Ce^{2t}$$

**Initial condition:**  $y(0) = 1$ :

$$1 = \frac{0}{2} - \frac{7}{4} + C \Rightarrow C = 1 + \frac{7}{4} = \frac{11}{4}$$

**Particular solution:**

$$y = \frac{t}{2} - \frac{7}{4} + \frac{11}{4}e^{2t}$$

**Behavior as  $t \rightarrow \infty$ :** Since  $e^{2t}$  grows exponentially,  $y(t) \rightarrow +\infty$  as  $t \rightarrow \infty$ . □

**Example 2.1.5** (Homogeneous Linear Equation)

Solve  $y' + 3y = 0$  with  $y(0) = 2$ .

*Solution.* For a homogeneous equation  $y' + p(t)y = 0$ , we can solve by separation:

$$\frac{dy}{y} = -p(t)dt$$

Integrating:  $\ln |y| = -\int p(t)dt + C$

With  $p(t) = 3$ :

$$\ln |y| = -3t + C \Rightarrow y = Ae^{-3t}$$

Using  $y(0) = 2$ :  $A = 2$

Therefore:  $y = 2e^{-3t}$  □

**Remark 2.1.6** (Structure of Solutions). *The general solution of a first-order linear equation  $y' + p(t)y = g(t)$  has the form:*

$$y = y_h + y_p$$

where:

- $y_h = Ce^{-\int p(t)dt}$  is the general solution to the homogeneous equation
- $y_p$  is any particular solution to the nonhomogeneous equation

As  $t \rightarrow \infty$ :

- If  $p(t) > 0$ , then  $y_h \rightarrow 0$  (transient behavior)
- If  $p(t) < 0$ , then  $y_h$  grows exponentially
- The long-term behavior is determined by  $y_p$  when  $p(t) > 0$

## 2.2 Separable Equations

**Definition 2.2.1** (Separable Equation)

A first-order ODE is **separable** if it has the form  $\frac{dy}{dt} = f(t)g(y)$ .

**Fact 2.2.1** (Solving Separable Equations)

Separate:  $\frac{dy}{g(y)} = f(t) dt$ , then integrate both sides. Check  $g(y) = 0$  separately (equilibrium solutions can be lost when dividing).

### Example 2.2.2

Solve  $\frac{dy}{dt} = \frac{11}{8}e^{-t/3}$  with  $y(0) = -1$ .

*Solution.* Already separated. Integrate:  $y = \frac{11}{8}(-3)e^{-t/3} + C = -\frac{33}{8}e^{-t/3} + C$ .

$$y(0) = -1: -1 = -\frac{33}{8} + C, \text{ so } C = \frac{25}{8}.$$

$$y(t) = -\frac{33}{8}e^{-t/3} + \frac{25}{8}$$

□

### Example 2.2.3

Solve  $\frac{dy}{dt} = \frac{3t^2+1}{2y}$ .

*Solution.* Separate:  $2y \, dy = (3t^2 + 1) \, dt$ . Integrate:  $y^2 = t^3 + t + C$ , so  $y = \pm\sqrt{t^3 + t + C}$ .

□

### Example 2.2.4

Solve  $\frac{dy}{dt} = \frac{t^2}{1+y^2}$ ,  $y(0) = 1$ .

*Solution.*  $(1 + y^2) \, dy = t^2 \, dt$ . Integrate:  $y + \frac{y^3}{3} = \frac{t^3}{3} + C$ .

$$y(0) = 1: 1 + \frac{1}{3} = 0 + C, \text{ so } C = \frac{4}{3}. \text{ The solution is given implicitly by } y + \frac{y^3}{3} = \frac{t^3}{3} + \frac{4}{3}.$$

□

### Example 2.2.5

Solve  $y' = -2ty$ ,  $y(0) = 3$ .

*Solution.*  $\frac{dy}{y} = -2t \, dt$ . Integrate:  $\ln |y| = -t^2 + C$ , so  $y = Ae^{-t^2}$ . With  $y(0) = 3$ :  $A = 3$ .

$$y(t) = 3e^{-t^2}$$

□

## 2.3 Modeling with First Order Differential Equations

**Fact 2.3.1 (Common Models)** • **Exponential growth/decay:**  $y' = ky \Rightarrow y = y_0e^{kt}$ .

- **Newton's Cooling:**  $T' = -k(T - T_{\text{env}})$ .
- **Mixing:**  $Q' = \text{rate in} - \text{rate out}$ .
- **Logistic:**  $P' = rP(1 - P/K)$ ,  $K = \text{carrying capacity}$ .

### Example 2.3.2 (Mixing Problem)

A 100-gal tank initially contains 50 lb of salt dissolved in water. Brine with 2 lb/gal flows in at 3 gal/min, and the well-stirred mixture flows out at 3 gal/min. Find the amount of salt  $Q(t)$  at time  $t$ .

*Solution.* Rate in =  $2 \cdot 3 = 6$  lb/min. Rate out =  $\frac{Q}{100} \cdot 3 = \frac{3Q}{100}$  lb/min.

$$Q' = 6 - \frac{3Q}{100}, \quad Q(0) = 50$$

This is linear:  $Q' + \frac{3}{100}Q = 6$ . Integrating factor  $\mu = e^{3t/100}$ .

$$\frac{d}{dt}[e^{3t/100}Q] = 6e^{3t/100} \Rightarrow e^{3t/100}Q = 200e^{3t/100} + C$$

So  $Q = 200 + Ce^{-3t/100}$ . Using  $Q(0) = 50$ :  $C = -150$ .

$$Q(t) = 200 - 150e^{-3t/100}$$

□

## 2.4 Differences Between Linear and Nonlinear Equations

### Theorem 2.4.1 (Existence and Uniqueness Theorem for First-Order Linear Equations)

If the functions  $p$  and  $g$  are continuous on an open interval  $I : \alpha < t < \beta$  containing the point  $t = t_0$ , then there exists a unique function  $y = \phi(t)$  that satisfies the differential equation

$$y' + p(t)y = g(t)$$

for each  $t$  in  $I$ , and that also satisfies the initial condition

$$y(t_0) = y_0$$

where  $y_0$  is an arbitrary prescribed initial value.

### Theorem 2.4.2 (Existence and Uniqueness Theorem for First-Order Linear Equations)

Let the functions  $f$  and  $\frac{\partial f}{\partial y}$  be continuous in some rectangle  $\alpha < t < \beta$ ,  $\gamma < y < \delta$  containing the point  $(t_0, y_0)$ . Then, in some interval  $t_0 - h < t < t_0 + h$  contained in  $\alpha < t < \beta$ , there is a unique solution  $y = \phi(t)$  of the initial value problem

$$y' = f(t, y), \quad y(t_0) = y_0.$$

### Example 2.4.3

Find an interval in which the initial value problem

$$ty' + 2y = 4t^2, \quad y(1) = 2$$

has a unique solution. Then do the same when the initial value is changed to  $y(-1) = 2$ .

*Solution.* Standard form:  $y' + \frac{2}{t}y = 4t$ . Here  $p(t) = \frac{2}{t}$  and  $g(t) = 4t$ . The only discontinuity is at  $t = 0$ .

For  $y(1) = 2$ :  $t_0 = 1$  lies in  $(0, \infty)$ , so the solution is guaranteed on  $(0, \infty)$ .

For  $y(-1) = 2$ :  $t_0 = -1$  lies in  $(-\infty, 0)$ , so the solution is guaranteed on  $(-\infty, 0)$ .  $\square$

### Example 2.4.4

Find an interval in which the initial value problem

$$\frac{dy}{dt} = \frac{3x^2 + 4x + 2}{2(y-1)}, \quad y(0) = -1$$

has a unique solution. Then do the same when the initial value is changed to  $y(0) = 1$ .

*Solution.* Here  $f(t, y) = \frac{3t^2+4t+2}{2(y-1)}$  and  $\frac{\partial f}{\partial y} = -\frac{3t^2+4t+2}{2(y-1)^2}$ . Both are continuous as long as  $y \neq 1$ .

For  $y(0) = -1$ : The point  $(0, -1)$  has  $y \neq 1$ , so  $f$  and  $f_y$  are continuous in a rectangle around it. A unique solution exists.

For  $y(0) = 1$ : At  $(0, 1)$ ,  $f$  is undefined (division by zero). The theorem does not guarantee existence or uniqueness.  $\square$

### Example 2.4.5

Consider the initial value problem

$$y' = y^{\frac{1}{3}}, \quad y(0) = 0$$

*Solution.*  $f(t, y) = y^{1/3}$  is continuous, but  $f_y = \frac{1}{3}y^{-2/3}$  blows up at  $y = 0$ . Theorem doesn't apply.

In fact, both  $y(t) = 0$  and  $y(t) = (\frac{2t}{3})^{3/2}$  (for  $t \geq 0$ ) satisfy the IVP — non-uniqueness!  $\square$

### Example 2.4.6

Determine the longest interval of existence for  $(t^2 - 9)y'' + \frac{1}{t}y' + (t - 1)y = 0$ ,  $y(2) = 1$ ,  $y'(2) = 0$ .

*Solution.* Standard form: divide by  $(t^2 - 9)$ . Singularities at  $t = \pm 3$  and  $t = 0$  (from  $1/t$ ). The initial point  $t_0 = 2$  lies in  $(0, 3)$ .  $\square$

## 2.5 Autonomous Equations and Population Dynamics

### Definition 2.5.1 (Autonomous Equation)

$y' = f(y)$  is **autonomous** if  $f$  does not depend on  $t$ . A constant solution  $y(t) = y^*$  with  $f(y^*) = 0$  is an **equilibrium**.

### Fact 2.5.1 (Stability (Phase Line))

For  $y' = f(y)$  with equilibrium  $y^*$ :

- **Stable**: arrows point toward  $y^*$  from both sides ( $f'(y^*) < 0$ ).
- **Unstable**: arrows point away ( $f'(y^*) > 0$ ).
- **Semi-stable**: arrows point the same direction on both sides.

### Example 2.5.2

For  $y' = y(3 - y)$ , find and classify all equilibria.

*Solution.* Set  $y(3 - y) = 0$ : equilibria at  $y = 0$  and  $y = 3$ .

$f(y) = 3y - y^2$ , so  $f'(y) = 3 - 2y$ . At  $y = 0$ :  $f'(0) = 3 > 0$  (unstable). At  $y = 3$ :  $f'(3) = -3 < 0$  (stable).

Alternatively: for  $0 < y < 3$ ,  $f(y) > 0$  (solutions increase); for  $y > 3$ ,  $f(y) < 0$  (solutions decrease). So  $y = 3$  is stable,  $y = 0$  is unstable.  $\square$

## 2.6 Exact Equation and Integrating Factors

### Definition 2.6.1 (Exact Differential Equation)

A first-order differential equation of the form

$$M(x, y) + N(x, y)y' = 0$$

or equivalently written as

$$M(x, y)dx + N(x, y)dy = 0$$

is called an **exact differential equation** if there exists a function  $\psi(x, y)$  such that''

$$\frac{\partial \psi}{\partial x} = M(x, y) \tag{1}$$

$$\frac{\partial \psi}{\partial y} = N(x, y) \tag{2}$$

**Theorem 2.6.1** (Test for Exactness)

Let the functions  $M$ ,  $N$ ,  $M_y$ , and  $N_x$ , where subscripts denote partial derivatives, be continuous in the rectangular region  $R : \alpha < x < \beta, \gamma < y < \delta$ . Then equation

$$M(x, y) + N(x, y)y' = 0$$

is an exact differential equation in  $R$  if and only if

$$M_y(x, y) = N_x(x, y)$$

at each point of  $R$ . That is, there exists a function  $\psi$  satisfying equations

$$\psi_x(x, y) = M(x, y), \quad \psi_y(x, y) = N(x, y),$$

## 3 Second Order Linear Differential Equations

### 3.1 Homogeneous Equations with Constant Coefficients

#### Definition 3.1.1

A **second-order linear homogeneous differential equation with constant coefficients** has the general form:

$$ay'' + by' + cy = 0$$

where  $a$ ,  $b$ , and  $c$  are real constants with  $a \neq 0$ .

Since  $(e^{rt})' = re^{rt}$  and  $(e^{rt})'' = r^2e^{rt}$ , exponentials are natural candidates for solutions.

#### Theorem 3.1.1 (Characteristic Equation Method)

To solve the differential equation  $ay'' + by' + cy = 0$ :

1. **Assume an exponential solution:** Let  $y = e^{rt}$  where  $r$  is a constant to be determined.
2. **Compute derivatives:**

$$\begin{aligned}y' &= re^{rt} \\y'' &= r^2e^{rt}\end{aligned}$$

3. **Substitute into the differential equation:**

$$a(r^2e^{rt}) + b(re^{rt}) + c(e^{rt}) = 0$$

4. **Factor out  $e^{rt}$  (which is never zero):**

$$e^{rt}(ar^2 + br + c) = 0$$

5. **Obtain the characteristic equation:**

$$ar^2 + br + c = 0$$

6. **Solve using the quadratic formula:**

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The discriminant  $\Delta = b^2 - 4ac$  determines the solution type:

**Theorem 3.1.2** (General Solutions Based on Root Types)

Let  $r_1$  and  $r_2$  be the roots of the characteristic equation  $ar^2 + br + c = 0$ . Then:

**Case 1: Real and Distinct Roots** ( $\Delta > 0$ )

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

where  $c_1$  and  $c_2$  are arbitrary constants determined by initial conditions.

**Case 2: Complex Conjugate Roots** ( $\Delta < 0$ ) If  $r_{1,2} = \alpha \pm \beta i$  where  $\alpha = -\frac{b}{2a}$  and  $\beta = \frac{\sqrt{4ac-b^2}}{2a}$ :

$$y(t) = e^{\alpha t} (c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

**Case 3: Repeated Real Root** ( $\Delta = 0$ ) If  $r_1 = r_2 = r = -\frac{b}{2a}$ :

$$y(t) = (c_1 + c_2 t) e^{rt}$$

The  $t$  factor is needed for linear independence.

**Remark 3.1.3.** *The physical interpretation varies by case:*

- **Distinct real roots:** *Exponential growth/decay (no oscillation)*
- **Complex roots:** *Damped or growing oscillations*
- **Repeated roots:** *Critical damping (fastest approach to equilibrium without oscillation)*

### Example 3.1.4

Consider the differential equation  $y'' - 3y' + 2y = 0$ .

**Step 1:** Form the characteristic equation:

$$r^2 - 3r + 2 = 0$$

**Step 2:** Factor or use quadratic formula:

$$(r - 1)(r - 2) = 0$$

So  $r_1 = 1$  and  $r_2 = 2$ .

**Step 3:** Since we have real and distinct roots, the general solution is:

$$y(t) = c_1 e^t + c_2 e^{2t}$$

**Verification:** We can verify this solution by substitution:

$$y = c_1 e^t + c_2 e^{2t}$$

$$y' = c_1 e^t + 2c_2 e^{2t}$$

$$y'' = c_1 e^t + 4c_2 e^{2t}$$

Substituting:  $y'' - 3y' + 2y = (c_1 e^t + 4c_2 e^{2t}) - 3(c_1 e^t + 2c_2 e^{2t}) + 2(c_1 e^t + c_2 e^{2t}) = 0$

### Example 3.1.5

Solve  $y'' + 5y' + 6y = 0$ ,  $y(0) = 2$ ,  $y'(0) = 3$ .

*Solution.*  $r^2 + 5r + 6 = (r + 2)(r + 3) = 0$ , so  $r = -2, -3$ . General solution:  $y = c_1 e^{-2t} + c_2 e^{-3t}$ .

$$y(0) = c_1 + c_2 = 2. \quad y'(0) = -2c_1 - 3c_2 = 3.$$

From the first:  $c_2 = 2 - c_1$ . Substituting:  $-2c_1 - 3(2 - c_1) = 3 \Rightarrow c_1 - 6 = 3 \Rightarrow c_1 = 9$ ,  $c_2 = -7$ .

$$y(t) = 9e^{-2t} - 7e^{-3t}$$

□

## 3.2 Solutions of Linear Homogeneous Equations; the Wronskian

### Definition 3.2.1 (Wronskian)

The Wronskian of two functions  $f$  and  $g$  is defined as

$$W(f, g)(t) = f(t)g'(t) - g(t)f'(t)$$

### Example 3.2.1

Let  $f(t) = e^t$  and  $g(t) = e^{2t}$ . Then we have:

$$W(f, g)(t) = e^t(2e^{2t}) - e^{2t}(e^t) = 2e^{3t} - e^{3t} = e^{3t}$$

Since  $W(f, g)(t) \neq 0$  for all  $t$ , the functions  $f$  and  $g$  are linearly independent.

### Theorem 3.2.2

If  $f$  and  $g$  are differentiable functions, on an open interval  $I$ , and if  $W(f, g)(t_0) \neq 0$ , for some point  $t_0 \in I$ , then  $f$  and  $g$  are linearly independent on  $I$ . Moreover, if  $f$  and  $g$  are linearly dependent on  $I$ , then the Wronskian  $W(f, g)(t) = 0$  for every  $t \in I$ .

### Theorem 3.2.3 (Abel's Theorem)

If  $y_1$  and  $y_2$  are two solutions of the second-order linear homogeneous differential equation

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

on an interval  $I$ , then the Wronskian  $W(y_1, y_2)(t)$  satisfies

$$W(y_1, y_2)(t) = W(y_1, y_2)(t_0) \exp \left[ - \int_{t_0}^t p(s) ds \right]$$

for any  $t_0 \in I$ .

### Theorem 3.2.4

Let  $y_1$  and  $y_2$  be the solutions of the second-order linear homogeneous differential equation

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

where  $p$  and  $q$  are continuous functions on an interval  $I$ . Then the Wronskian  $W(y_1, y_2)(t)$  satisfies

$$W(y_1, y_2)(t) = W(y_1, y_2)(t_0) \exp \left[ - \int_{t_0}^t p(s) ds \right]$$

for any  $t_0 \in I$ .

## 3.3 Complex Roots of the Characteristic Equation

We continue our discussion of the second-order linear differential equation

$$ay'' + by' + cy = 0$$

where  $a$ ,  $b$ , and  $c$  are constants. Here we the characteristic equation is

$$ar^2 + br + c = 0$$

and the general solution is

$$y(t) = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

where  $\alpha = -\frac{b}{2a}$  and  $\beta = \frac{\sqrt{4ac-b^2}}{2a}$ . However,  $b^2 - 4ac$  could be negative, leading to complex roots. Assuming that they are complex, we can write the roots as

$$r_1 = \lambda + i\mu, \quad r_2 = \lambda - i\mu$$

where  $\lambda$  and  $\mu$  are real. Therefore the expressions for  $y$  are

$$y(t) = e^{\lambda t}(c_1 \cos(\mu t) + c_2 \sin(\mu t))$$

### Example 3.3.1

Find the general solution of the differential equation

$$y'' + y' + 9.25y = 0.$$

Also find the solution that satisfies the initial conditions  $y(0) = 2$  and  $y'(0) = 0.8$ .

*Solution.* Characteristic equation:  $r^2 + r + 9.25 = 0$ .

$$r = \frac{-1 \pm \sqrt{1 - 37}}{2} = \frac{-1 \pm \sqrt{-36}}{2} = \frac{-1 \pm 6i}{2} = -\frac{1}{2} \pm 3i$$

So  $\lambda = -\frac{1}{2}$ ,  $\mu = 3$ . General solution:

$$y(t) = e^{-t/2}(c_1 \cos 3t + c_2 \sin 3t)$$

Apply  $y(0) = 2$ :  $c_1 = 2$ .

$$y'(t) = -\frac{1}{2}e^{-t/2}(c_1 \cos 3t + c_2 \sin 3t) + e^{-t/2}(-3c_1 \sin 3t + 3c_2 \cos 3t).$$

$$\text{Apply } y'(0) = 0.8: -\frac{1}{2}(2) + 3c_2 = 0.8 \Rightarrow -1 + 3c_2 = 0.8 \Rightarrow c_2 = 0.6.$$

$$y(t) = e^{-t/2}(2 \cos 3t + 0.6 \sin 3t)$$

□

### Example 3.3.2

Find the solution of the initial value problem

$$16y'' - 8y' + 145y = 0, \quad y(0) = -2, \quad y'(0) = 1.$$

*Solution.* Characteristic equation:  $16r^2 - 8r + 145 = 0$ .

$$r = \frac{8 \pm \sqrt{64 - 9280}}{32} = \frac{8 \pm \sqrt{-9216}}{32} = \frac{8 \pm 96i}{32} = \frac{1}{4} \pm 3i$$

General solution:  $y(t) = e^{t/4}(c_1 \cos 3t + c_2 \sin 3t)$ .

$$y(0) = c_1 = -2.$$

$$y'(t) = \frac{1}{4}e^{t/4}(c_1 \cos 3t + c_2 \sin 3t) + e^{t/4}(-3c_1 \sin 3t + 3c_2 \cos 3t).$$

$$y'(0) = \frac{1}{4}c_1 + 3c_2 = \frac{1}{4}(-2) + 3c_2 = -\frac{1}{2} + 3c_2 = 1 \Rightarrow c_2 = \frac{1}{2}.$$

$$y(t) = e^{t/4} \left( -2 \cos 3t + \frac{1}{2} \sin 3t \right)$$

□

### Example 3.3.3

Find the general solution of the differential equation

$$y'' + 9y = 0.$$

*Solution.* Characteristic equation:  $r^2 + 9 = 0 \Rightarrow r = \pm 3i$ . Here  $\lambda = 0$  and  $\mu = 3$ .

$$y(t) = c_1 \cos 3t + c_2 \sin 3t$$

This is pure oscillation (no damping) since  $\lambda = 0$ .

□

## 3.4 Repeated Roots; Reduction of Order

We know how to solve the equation

$$ay'' + by' + cy = 0 \tag{1}$$

when the roots of the characteristic equation

$$ar^2 + br + c = 0 \tag{2}$$

either are real and different or are complex conjugates. Now we consider the third possibility, namely, that the two roots  $r_1$  and  $r_2$  are equal. This case is transitional between the other two and occurs when the discriminant  $b^2 - 4ac$  is zero. Then it follows from the quadratic formula that

$$r_1 = r_2 = -\frac{b}{2a}. \tag{3}$$

The difficulty is immediately apparent; both roots yield the same solution

$$y_1(t) = e^{-bt/(2a)} \tag{4}$$

of the differential equation (1), and it is not obvious how to find a second solution.

For the polynomial  $ay'' + by' + cy = 0$  the characteristic equation is

$$ar^2 + br + c = 0$$

If the roots are repeated, i.e.,  $r_1 = r_2 = r$ , then the general solution is

$$y(t) = (c_1 + c_2t)e^{rt}$$

### Example 3.4.1

Solve the differential equation

$$y'' + 4y' + 4y = 0$$

*Solution.* Characteristic equation:  $r^2 + 4r + 4 = 0 \Rightarrow (r + 2)^2 = 0 \Rightarrow r = -2$  (repeated).

$$y(t) = c_1e^{-2t} + c_2te^{-2t} = (c_1 + c_2t)e^{-2t}$$

□

### Example 3.4.2

Find the solution of the initial value problem

$$y'' - y' + \frac{1}{4}y = 0, \quad y(0) = 2, \quad y'(0) = \frac{1}{3}.$$

*Solution.* Characteristic equation:  $r^2 - r + \frac{1}{4} = 0 \Rightarrow (r - \frac{1}{2})^2 = 0 \Rightarrow r = \frac{1}{2}$  (repeated).

General solution:  $y(t) = (c_1 + c_2t)e^{t/2}$ .

$$y(0) = c_1 = 2.$$

$$y'(t) = c_2e^{t/2} + \frac{1}{2}(c_1 + c_2t)e^{t/2}, \text{ so } y'(0) = c_2 + \frac{1}{2}c_1 = c_2 + 1 = \frac{1}{3} \Rightarrow c_2 = -\frac{2}{3}.$$

$$y(t) = \left(2 - \frac{2}{3}t\right)e^{t/2}$$

□

## 3.5 Nonhomogeneous Equations; Method of Undetermined Coefficients

Now, we could have a situation where the differential equation might also be non-homogeneous which is

$$L[y] = y'' + p(t)y' + q(t)y = g(t)$$

where  $p, q,$  and  $g$  are given (continuous) functions on the open interval  $I$ . The equation

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

is very useful in solving this problem.

### Theorem 3.5.1

If  $Y_1$  and  $Y_2$  are two solutions of the non-homogeneous linear differential equation  $L[y] = y'' + p(t)y' + q(t)y = g(t)$ , then their difference  $Y_1 - Y_2$  is a solution of the corresponding homogeneous differential equation (2). If, in addition,  $y_1$  and  $y_2$  form a fundamental set of solutions of equation  $L[y] = y'' + p(t)y' + q(t)y = 0$ , then

$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t)$$

where  $c_1$  and  $c_2$  are certain constants.

### Theorem 3.5.2

The general solution for the non-homogeneous equation  $L[y] = y'' + p(t)y' + q(t)y = g(t)$  can be expressed as

$$y = \phi(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

where  $y_1$  and  $y_2$  form a fundamental set of solutions of the corresponding homogeneous equation  $L[y] = y'' + p(t)y' + q(t)y = 0$  and where  $Y$  is any particular solution of the non-homogeneous equation.

### Fact 3.5.3 (Method of Undetermined Coefficients)

For  $ay'' + by' + cy = g(t)$  with constant coefficients, guess  $y_p(t)$  based on the form of  $g(t)$ :

$g(t)$	Guess $y_p(t)$
$P_n(t)$ (polynomial of degree $n$ )	$A_n t^n + \dots + A_1 t + A_0$
$e^{\alpha t}$	$A e^{\alpha t}$
$\cos(\beta t)$ or $\sin(\beta t)$	$A \cos(\beta t) + B \sin(\beta t)$
$P_n(t)e^{\alpha t}$	$(A_n t^n + \dots + A_0)e^{\alpha t}$
$e^{\alpha t} \cos(\beta t)$ or $e^{\alpha t} \sin(\beta t)$	$e^{\alpha t}(A \cos \beta t + B \sin \beta t)$

**Resonance rule:** If any term in  $y_p$  is a solution of the homogeneous equation, multiply  $y_p$  by  $t$  (or  $t^2$  for double roots) until no term is a homogeneous solution.

**Superposition:** If  $g(t) = g_1(t) + g_2(t)$ , find particular solutions for each separately and add.

### Example 3.5.4

Find the form of a particular solution for  $y'' + y' + y = \sin^2 t$ .

*Solution.* Rewrite:  $\sin^2 t = \frac{1}{2} - \frac{1}{2} \cos(2t)$ .

The homogeneous equation  $y'' + y' + y = 0$  has roots  $r = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ , so no resonance with the constant or  $\cos(2t)$  terms.

For  $g_1 = \frac{1}{2}$ : guess  $y_{p1} = A$ .

For  $g_2 = -\frac{1}{2} \cos(2t)$ : guess  $y_{p2} = B \cos(2t) + C \sin(2t)$ .

Total:  $y_p(t) = A + B \cos(2t) + C \sin(2t)$ . □

**Example 3.5.5**Solve  $y'' + 3y' + 2y = e^{-t}$ .

*Solution.* Homogeneous:  $r^2 + 3r + 2 = (r + 1)(r + 2) = 0$ , so  $y_h = c_1e^{-t} + c_2e^{-2t}$ .

Guess  $y_p = Ae^{-t}$ . But  $e^{-t}$  is already a homogeneous solution! Multiply by  $t$ : try  $y_p = Ate^{-t}$ .

$$y'_p = Ae^{-t} - Ate^{-t}, \quad y''_p = -2Ae^{-t} + Ate^{-t}.$$

Substituting:  $(-2A + At)e^{-t} + 3(A - At)e^{-t} + 2Ate^{-t} = e^{-t}$ .

Collecting:  $(-2A + 3A)e^{-t} + (A - 3A + 2A)te^{-t} = Ae^{-t} = e^{-t}$ . So  $A = 1$ .

$$y = c_1e^{-t} + c_2e^{-2t} + te^{-t}$$

□

**Example 3.5.6**Solve  $y'' - 4y = 3t + \sin t$ .

*Solution.* Homogeneous:  $r^2 - 4 = 0$ , so  $r = \pm 2$ ,  $y_h = c_1e^{2t} + c_2e^{-2t}$ .

For  $g_1 = 3t$ : guess  $y_{p1} = At + B$ . Substituting:  $-4(At + B) = 3t \Rightarrow A = -3/4, B = 0$ .

For  $g_2 = \sin t$ : guess  $y_{p2} = C \cos t + D \sin t$ . Substituting:  $(-C \cos t - D \sin t) - 4(C \cos t + D \sin t) = \sin t$ , so  $-5C = 0$  and  $-5D = 1$ . Thus  $C = 0, D = -1/5$ .

$$y = c_1e^{2t} + c_2e^{-2t} - \frac{3}{4}t - \frac{1}{5}\sin t$$

□

**Fact 3.5.7 (Variation of Parameters)**

For  $y'' + p(t)y' + q(t)y = g(t)$  with fundamental set  $y_1, y_2$ :

$$y_p(t) = -y_1 \int \frac{y_2 g}{W} dt + y_2 \int \frac{y_1 g}{W} dt$$

Works for any  $g(t)$ , not just polynomials/exponentials/trig.

## 6 The Laplace Transform

### 6.1 Definition of the Laplace Transform

**Definition 6.1.1** (Laplace Transform)

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t) dt, \text{ provided the integral converges.}$$

**Fact 6.1.1** (Table of Common Laplace Transforms)

$f(t)$	$F(s) = \mathcal{L}\{f\}$
1	$\frac{1}{s}, \quad s > 0$
$e^{at}$	$\frac{1}{s-a}, \quad s > a$
$t^n \quad (n = 1, 2, \dots)$	$\frac{n!}{s^{n+1}}$
$\sin(bt)$	$\frac{b}{s^2 + b^2}$
$\cos(bt)$	$\frac{s}{s^2 + b^2}$
$e^{at} \sin(bt)$	$\frac{b}{(s-a)^2 + b^2}$
$e^{at} \cos(bt)$	$\frac{s-a}{(s-a)^2 + b^2}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$u_c(t)$ (step function)	$\frac{e^{-cs}}{s}$
$\delta(t-c)$ (impulse)	$e^{-cs}$

**Fact 6.1.2** (Key Properties) 1. **Linearity:**  $\mathcal{L}\{af + bg\} = a\mathcal{L}\{f\} + b\mathcal{L}\{g\}$ .

2. **First shifting theorem** ( $s$ -shift):  $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$ .

3. **Derivatives:**  $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$  and  $\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$ .

4. **Second shifting theorem** ( $t$ -shift):  $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s)$ .

5. **Transform of  $t \cdot f(t)$ :**  $\mathcal{L}\{tf(t)\} = -F'(s)$ .

### 6.2 Solution of Initial Value Problems

**Fact 6.2.1 (Solving IVPs with Laplace)**

For  $ay'' + by' + cy = g(t)$ ,  $y(0) = y_0$ ,  $y'(0) = y'_0$ :

1. Transform both sides:  $a[s^2Y - sy_0 - y'_0] + b[sY - y_0] + cY = G(s)$ .
2. Solve for  $Y(s)$ .
3. Invert using partial fractions and the table.

**Example 6.2.2**

Solve  $y'' - y' - 2y = 0$ ,  $y(0) = 1$ ,  $y'(0) = 0$  using Laplace transforms.

*Solution.* Take  $\mathcal{L}$ :  $[s^2Y - s - 0] - [sY - 1] - 2Y = 0$ .

$$(s^2 - s - 2)Y = s - 1 \Rightarrow Y(s) = \frac{s - 1}{s^2 - s - 2} = \frac{s - 1}{(s - 2)(s + 1)}$$

Partial fractions:  $\frac{s-1}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1}$ .  
 $s = 2$ :  $\frac{1}{3} = A$ .  $s = -1$ :  $-\frac{2}{3} = B = \frac{2}{3}$ .

$$Y(s) = \frac{1/3}{s-2} + \frac{2/3}{s+1} \Rightarrow y(t) = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$$

□

**Example 6.2.3**

Solve  $y'' + 4y = \sin t$ ,  $y(0) = 0$ ,  $y'(0) = 0$ .

*Solution.* Transform:  $(s^2 + 4)Y = \frac{1}{s^2 + 1}$ , so  $Y = \frac{1}{(s^2 + 1)(s^2 + 4)}$ .

Partial fractions:  $\frac{1}{(s^2 + 1)(s^2 + 4)} = \frac{1}{3} \cdot \frac{1}{s^2 + 1} - \frac{1}{3} \cdot \frac{1}{s^2 + 4}$ .

Invert:  $y(t) = \frac{1}{3} \sin t - \frac{1}{6} \sin 2t$ .

□

**Example 6.2.4**

Solve  $y' + 2y = e^{-t}$ ,  $y(0) = 3$ .

*Solution.*  $(sY - 3) + 2Y = \frac{1}{s+1}$ , so  $(s+2)Y = 3 + \frac{1}{s+1} = \frac{3s+4}{s+1}$ .

$Y = \frac{3s+4}{(s+1)(s+2)}$ . Partial fractions:  $\frac{3s+4}{(s+1)(s+2)} = \frac{1}{s+1} + \frac{2}{s+2}$ .

$y(t) = e^{-t} + 2e^{-2t}$ .

□

## 6.3 Step Functions

**Definition 6.3.1** (Heaviside Step Function)

The **unit step function** (Heaviside function) is

$$u_c(t) = \begin{cases} 0 & \text{if } t < c \\ 1 & \text{if } t \geq c \end{cases}$$

Its Laplace transform is  $\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}$ .

**Fact 6.3.1** (Writing Piecewise Functions with Step Functions)

A piecewise function  $f(t) = \begin{cases} f_1(t), & 0 \leq t < a \\ f_2(t), & a \leq t < b \\ f_3(t), & t \geq b \end{cases}$  can be written as:

$$f(t) = f_1(t) + [f_2(t) - f_1(t)]u_a(t) + [f_3(t) - f_2(t)]u_b(t)$$

## 6.4 Differential Equations with Discontinuous Forcing Functions

**Fact 6.4.1**

Express  $g(t)$  using step functions, transform using  $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s)$ , solve for  $Y(s)$ , and invert.

**Example 6.4.2**

Solve  $y'' + y = g(t)$ ,  $y(0) = 0$ ,  $y'(0) = 0$  where  $g(t) = \begin{cases} 1 & 0 \leq t < \pi \\ 0 & t \geq \pi \end{cases}$ .

*Solution.* Write  $g(t) = 1 - u_\pi(t)$ . Take  $\mathcal{L}$ :

$$(s^2 + 1)Y = \frac{1}{s} - \frac{e^{-\pi s}}{s} \Rightarrow Y = \frac{1}{s(s^2 + 1)} - \frac{e^{-\pi s}}{s(s^2 + 1)}$$

Partial fractions:  $\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$ .

So  $y(t) = (1 - \cos t) - u_\pi(t)(1 - \cos(t - \pi)) = (1 - \cos t) - u_\pi(t)(1 + \cos t)$ . □

## 6.5 Impulse Functions

**Definition 6.5.1** (Dirac Delta)

$\delta(t - t_0)$  satisfies  $\int_{-\infty}^{\infty} \delta(t - t_0)f(t) dt = f(t_0)$ . It's an instantaneous impulse at  $t_0$ .

$$\mathcal{L}\{\delta(t - t_0)\} = e^{-t_0 s}, \quad \mathcal{L}\{\delta(t)\} = 1$$

### Example 6.5.1

Solve  $y'' + 2y' + 2y = \delta(t - \pi)$ ,  $y(0) = 0$ ,  $y'(0) = 0$ .

*Solution.* Take  $\mathcal{L}$ :  $(s^2 + 2s + 2)Y = e^{-\pi s}$ .

$$Y = \frac{e^{-\pi s}}{(s+1)^2 + 1}$$

Since  $\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} = e^{-t} \sin t$ , by the  $t$ -shift:

$$y(t) = u_{\pi}(t)e^{-(t-\pi)} \sin(t - \pi)$$

□

## 6.6 The Convolution Integral

### Definition 6.6.1 (Convolution)

The **convolution** of  $f$  and  $g$  is

$$(f * g)(t) = \int_0^t f(t - \tau)g(\tau) d\tau$$

### Theorem 6.6.1 (Convolution Theorem)

If  $F(s) = \mathcal{L}\{f\}$  and  $G(s) = \mathcal{L}\{g\}$ , then

$$\mathcal{L}\{f * g\} = F(s) \cdot G(s)$$

Equivalently:  $\mathcal{L}^{-1}\{F(s)G(s)\} = (f * g)(t)$ .

**Remark 6.6.2.** *Properties of convolution:*

- *Commutative:*  $f * g = g * f$
- *Associative:*  $f * (g * h) = (f * g) * h$
- *Distributive:*  $f * (g + h) = f * g + f * h$

## 7 Systems of First Order Linear Equations

### 7.1 Introduction

#### Definition 7.1.1

$\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t)$  is a **system of first-order linear ODEs**. Homogeneous when  $\mathbf{g} = \mathbf{0}$ .

#### Fact 7.1.1 (Converting to a System)

Set  $x_1 = y$ ,  $x_2 = y'$ ,  $\dots$ ,  $x_n = y^{(n-1)}$ . Then  $x'_k = x_{k+1}$  and the last equation uses the ODE to express  $x'_n$ .

#### Example 7.1.2

Convert  $u'' + \omega^2 u = 0$  to a first-order system.

*Solution.* Let  $x_1 = u$ ,  $x_2 = u'$ . Then  $x'_1 = x_2$  and  $x'_2 = -\omega^2 x_1$ .

$$\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \mathbf{x}$$

□

### 7.2 Matrices

#### Fact 7.2.1 (Essential Matrix Operations)

For  $n \times n$  matrices  $A$  and  $B$ :

- **Trace:**  $\text{tr}(A) = a_{11} + a_{22} + \dots + a_{nn}$  (sum of diagonal entries).
- **Determinant** (for  $2 \times 2$ ):  $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$ .
- **Inverse** (for  $2 \times 2$ ):  $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ , exists iff  $\det(A) \neq 0$ .

### 7.3 Systems of Linear Algebraic Equations; Linear Independence, Eigenvalues, Eigenvectors

**Definition 7.3.1** (Eigenvalues and Eigenvectors)

A scalar  $\lambda$  is an **eigenvalue** of the  $n \times n$  matrix  $A$  if there exists a nonzero vector  $\mathbf{v}$  such that

$$A\mathbf{v} = \lambda\mathbf{v}$$

The vector  $\mathbf{v}$  is called an **eigenvector** corresponding to  $\lambda$ .

**Fact 7.3.1** (Finding Eigenvalues)

Eigenvalues satisfy the **characteristic equation**:

$$\det(A - \lambda I) = 0$$

For a  $2 \times 2$  matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ :

$$\lambda^2 - \operatorname{tr}(A)\lambda + \det(A) = 0 \Rightarrow \lambda^2 - (a + d)\lambda + (ad - bc) = 0$$

**Fact 7.3.2** (Finding Eigenvectors)

For each eigenvalue  $\lambda$ , solve  $(A - \lambda I)\mathbf{v} = \mathbf{0}$  for  $\mathbf{v} \neq \mathbf{0}$ .

## 7.4 Basic Theory of Systems of First-Order Linear Equations

**Theorem 7.4.1** (Existence and Uniqueness for Systems)

If  $P(t)$  and  $\mathbf{g}(t)$  are continuous on an open interval  $I$  containing  $t_0$ , then there exists a unique solution  $\mathbf{x} = \phi(t)$  of

$$\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

throughout the interval  $I$ .

**Theorem 7.4.2** (Superposition Principle)

If  $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(n)}$  are solutions of the homogeneous system  $\mathbf{x}' = P(t)\mathbf{x}$ , then

$$\mathbf{x} = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \dots + c_n\mathbf{x}^{(n)}$$

is also a solution for any constants  $c_1, \dots, c_n$ .

**Definition 7.4.1** (Fundamental Set of Solutions)

The solutions  $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$  form a **fundamental set** if they are linearly independent, equivalently, if the Wronskian  $W = \det[\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}] \neq 0$ .

## 7.5 Homogeneous Linear Systems with Constant Coefficients

### Fact 7.5.1 (Eigenvalue Method for $\mathbf{x}' = A\mathbf{x}$ )

For a  $2 \times 2$  constant coefficient system:

1. Find eigenvalues  $\lambda_1, \lambda_2$  from  $\det(A - \lambda I) = 0$ .
2. Find corresponding eigenvectors  $\mathbf{v}_1, \mathbf{v}_2$ .
3. If eigenvalues are real and distinct, the general solution is:

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t} \mathbf{v}_1 + c_2 e^{\lambda_2 t} \mathbf{v}_2$$

### Example 7.5.2

Solve  $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}$ .

*Solution.* Characteristic equation:  $\lambda^2 - 2\lambda + 1 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) = 0$ .

$$\lambda_1 = 3: (A - 3I)\mathbf{v} = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \mathbf{v} = \mathbf{0} \Rightarrow \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

$$\lambda_2 = -1: (A + I)\mathbf{v} = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \mathbf{v} = \mathbf{0} \Rightarrow \mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

$$\mathbf{x}(t) = c_1 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

□

### Fact 7.5.3 (Stability from Eigenvalues)

$\mathbf{x}(t) \rightarrow \mathbf{0}$  as  $t \rightarrow \infty$  iff all eigenvalues have negative real parts. For  $2 \times 2$ :

$$\text{tr}(A) < 0 \quad \text{and} \quad \det(A) > 0$$

### Example 7.5.4

Solve  $\mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \mathbf{x}$ .

*Solution.*  $\det(A - \lambda I) = (3 - \lambda)(-1 - \lambda) + 8 = \lambda^2 - 2\lambda + 5 = 0$ .

$$\lambda = \frac{2 \pm \sqrt{4 - 20}}{2} = 1 \pm 2i. \text{ So } \alpha = 1, \beta = 2.$$

$$\text{For } \lambda = 1 + 2i: (A - (1 + 2i)I)\mathbf{v} = \begin{bmatrix} 2 - 2i & -2 \\ 4 & -2 - 2i \end{bmatrix} \mathbf{v} = \mathbf{0}.$$

From row 1:  $(2 - 2i)v_1 = 2v_2$ , pick  $\mathbf{v} = \begin{bmatrix} 1 \\ 1 - i \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ .

$\mathbf{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ,  $\mathbf{b} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ . Since  $\alpha = 1 > 0$ , this is an **unstable spiral**.

$$\mathbf{x}(t) = c_1 e^t \begin{bmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{bmatrix} + c_2 e^t \begin{bmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{bmatrix}$$

□

### Example 7.5.5

For  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , when does  $\mathbf{x}' = A\mathbf{x}$  have solutions approaching  $\mathbf{0}$ ?

*Solution.* Need  $\text{tr}(A) = a + d < 0$  and  $\det(A) = ad - bc > 0$ . Both conditions must hold.

□

## 7.6 Complex-Valued Eigenvalues

### Fact 7.6.1 (Complex Eigenvalue Method)

If  $A$  is a real  $2 \times 2$  matrix with complex eigenvalues  $\lambda = \alpha \pm \beta i$  and eigenvector  $\mathbf{v} = \mathbf{a} + i\mathbf{b}$  for  $\lambda = \alpha + \beta i$ , then two real-valued solutions are:

$$\mathbf{x}_1(t) = e^{\alpha t}(\mathbf{a} \cos \beta t - \mathbf{b} \sin \beta t)$$

$$\mathbf{x}_2(t) = e^{\alpha t}(\mathbf{a} \sin \beta t + \mathbf{b} \cos \beta t)$$

The general solution is  $\mathbf{x}(t) = c_1 \mathbf{x}_1(t) + c_2 \mathbf{x}_2(t)$ .

### Example 7.6.2

Solve  $\mathbf{x}' = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \mathbf{x}$ .

*Solution.* Characteristic equation:  $\lambda^2 + 2\lambda + 2 = 0 \Rightarrow \lambda = -1 \pm i$ . So  $\alpha = -1$ ,  $\beta = 1$ .

For  $\lambda = -1 + i$ :  $(A - \lambda I)\mathbf{v} = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \mathbf{v} = \mathbf{0} \Rightarrow \mathbf{v} = \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ .

So  $\mathbf{a} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ,  $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ .

$$\mathbf{x}(t) = c_1 e^{-t} \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

This is a stable spiral (since  $\alpha = -1 < 0$ ).

□

## 7.7 Fundamental Matrices

### Definition 7.7.1 (Fundamental Matrix)

$\Psi(t) = [\mathbf{x}^{(1)} \mid \cdots \mid \mathbf{x}^{(n)}]$  with linearly independent solution columns. Satisfies  $\Psi' = A\Psi$ ,  $\det(\Psi) \neq 0$ .

### Definition 7.7.2 (Matrix Exponential)

$e^{At} = I + At + \frac{(At)^2}{2!} + \cdots$  The unique fundamental matrix with  $e^{A \cdot 0} = I$ . IVP solution:  $\mathbf{x}(t) = e^{At}\mathbf{x}_0$ .

## 7.8 Repeated Eigenvalues

### Fact 7.8.1 (Repeated Eigenvalue $\lambda$ ( $2 \times 2$ case))

If  $A$  has a repeated eigenvalue  $\lambda$  with only one linearly independent eigenvector  $\mathbf{v}$ :

1. First solution:  $\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v}$ .
2. Find a **generalized eigenvector**  $\mathbf{w}$  by solving  $(A - \lambda I)\mathbf{w} = \mathbf{v}$ .
3. Second solution:  $\mathbf{x}_2(t) = te^{\lambda t}\mathbf{v} + e^{\lambda t}\mathbf{w}$ .

General solution:  $\mathbf{x}(t) = c_1e^{\lambda t}\mathbf{v} + c_2(te^{\lambda t}\mathbf{v} + e^{\lambda t}\mathbf{w})$ .

If there *are* two linearly independent eigenvectors (i.e.,  $A = \lambda I$ ), the general solution is simply  $\mathbf{x}(t) = e^{\lambda t}(c_1\mathbf{v}_1 + c_2\mathbf{v}_2)$ .

### Example 7.8.2

Solve  $\mathbf{x}' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \mathbf{x}$ .

*Solution.* Characteristic equation:  $\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0 \Rightarrow \lambda = 2$  (repeated).

Eigenvector:  $(A - 2I)\mathbf{v} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \mathbf{v} = \mathbf{0} \Rightarrow \mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ .

Only one eigenvector, so find  $\mathbf{w}$ :  $(A - 2I)\mathbf{w} = \mathbf{v}$ :  $\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \mathbf{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow -w_1 - w_2 = 1 \Rightarrow \mathbf{w} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ .

$$\mathbf{x}(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \left( te^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^{2t} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right)$$

□

## 7.9 Nonhomogeneous Linear Systems

**Fact 7.9.1** (Undetermined Coefficients for Systems)

For  $\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t)$ , the general solution is  $\mathbf{x} = \mathbf{x}_h + \mathbf{x}_p$  where  $\mathbf{x}_h$  is the general homogeneous solution and  $\mathbf{x}_p$  is a particular solution.

**Fact 7.9.2** (Variation of Parameters for Systems)

If  $\Psi(t)$  is a fundamental matrix for  $\mathbf{x}' = A\mathbf{x}$ , then a particular solution of  $\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t)$  is:

$$\mathbf{x}_p(t) = \Psi(t) \int \Psi^{-1}(t)\mathbf{g}(t) dt$$

## 9 Nonlinear Systems of Differential Equations

### 9.1 The Phase Plane: Linear Systems

**Fact 9.1.1** (Classification of Equilibria for  $\mathbf{x}' = A\mathbf{x}$  ( $2 \times 2$ ))

Let  $\lambda_1, \lambda_2$  be eigenvalues of  $A$  with  $\text{tr}(A) = \lambda_1 + \lambda_2$  and  $\det(A) = \lambda_1\lambda_2$ .

Eigenvalues	Type	Stability
$\lambda_1 < \lambda_2 < 0$ (real, same sign neg)	Node	Asymptotically stable
$0 < \lambda_1 < \lambda_2$ (real, same sign pos)	Node	Unstable
$\lambda_1 < 0 < \lambda_2$ (real, opposite signs)	Saddle point	Unstable
$\alpha \pm \beta i, \alpha < 0$	Spiral point	Asymptotically stable
$\alpha \pm \beta i, \alpha > 0$	Spiral point	Unstable
$\pm \beta i$ (pure imaginary)	Center	Stable (not asymptotically)
$\lambda_1 = \lambda_2 < 0$ , 2 eigenvectors	Proper/star node	Asymptotically stable
$\lambda_1 = \lambda_2 < 0$ , 1 eigenvector	Improper node	Asymptotically stable

**Remark 9.1.2** (Quick Classification via Trace and Determinant). For  $A$  with  $T = \text{tr}(A)$  and  $D = \det(A)$ :

- $D < 0$ : Saddle point (always unstable).
- $D > 0, T < 0$ : Stable (node if  $T^2 - 4D > 0$ , spiral if  $T^2 - 4D < 0$ ).
- $D > 0, T > 0$ : Unstable (node or spiral).
- $D > 0, T = 0$ : Center.
- $D = 0$ : Degenerate (non-isolated equilibria).

### 9.2 Autonomous Systems and Stability

**Definition 9.2.1**

$\mathbf{x}' = \mathbf{f}(\mathbf{x})$  is **autonomous** ( $\mathbf{f}$  doesn't depend on  $t$ ). Points where  $\mathbf{f}(\mathbf{x}_0) = \mathbf{0}$  are **equilibria**.

- **Stable**: nearby solutions stay nearby for all  $t \geq 0$ .
- **Asymptotically stable**: stable and  $\mathbf{x}(t) \rightarrow \mathbf{x}_0$  as  $t \rightarrow \infty$ .
- **Unstable**: not stable.

### 9.3 Almost Linear Systems

### Definition 9.3.1 (Linearization / Jacobian)

Near an equilibrium  $\mathbf{x}_0$  of  $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ , set  $\mathbf{u} = \mathbf{x} - \mathbf{x}_0$  and linearize:  $\mathbf{u}' \approx J(\mathbf{x}_0)\mathbf{u}$  where the **Jacobian** is

$$J = \begin{bmatrix} \partial f_1 / \partial x_1 & \partial f_1 / \partial x_2 \\ \partial f_2 / \partial x_1 & \partial f_2 / \partial x_2 \end{bmatrix}_{\mathbf{x}_0}$$

### Theorem 9.3.1 (Stability of Almost Linear Systems)

If eigenvalues of  $J(\mathbf{x}_0)$  have nonzero real parts, the nonlinear system has the same type/stability as the linearized one near  $\mathbf{x}_0$ .

**Exception:** pure imaginary eigenvalues ( $\pm\beta i$ ) predict a center, but the nonlinear system could be a spiral or center — further analysis needed.

### Example 9.3.2

Classify the equilibria of  $\mathbf{x}' = \begin{bmatrix} -1 & -1 \\ -\alpha & -1 \end{bmatrix} \mathbf{x}$ .

*Solution.* Eigenvalues:  $\lambda = -1 \pm \sqrt{\alpha}$ .

- $\alpha < 0$ :  $\lambda = -1 \pm i\sqrt{|\alpha|}$  (complex, negative real part)  $\Rightarrow$  **Stable spiral**.
- $0 < \alpha < 1$ :  $\lambda = -1 \pm \sqrt{\alpha}$ , both negative  $\Rightarrow$  **Stable node**.
- $\alpha = 1$ :  $\lambda = 0$  or  $\lambda = -2 \Rightarrow$  Degenerate case.
- $\alpha > 1$ :  $\sqrt{\alpha} > 1$ , so  $\lambda_1 > 0$  and  $\lambda_2 < 0 \Rightarrow$  **Unstable saddle**.
- $\alpha = 0$ :  $\lambda = -1$  (repeated)  $\Rightarrow$  **Stable improper node**.

Note: Saddle points are *never* stable. □

## 9.4 Competing Species

### Example 9.4.1

Find and classify all equilibria of  $x' = x - xy$ ,  $y' = -y + xy$ .

*Solution.* Equilibria:  $x(1 - y) = 0$  and  $y(-1 + x) = 0$ .

Case  $x = 0$ : then  $-y = 0$ , so  $(0, 0)$ . Case  $y = 1$ : then  $-1 + x = 0$ , so  $(1, 1)$ .

$$J = \begin{bmatrix} 1 - y & -x \\ y & -1 + x \end{bmatrix}.$$

At  $(0, 0)$ :  $J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ , eigenvalues  $1, -1$ . **Saddle** (unstable).

At  $(1, 1)$ :  $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ , eigenvalues  $\pm i$ . **Center** for linearized system. □

**Definition 9.4.1 (Competing Species)**

$$x' = x(\alpha_1 - \beta_1 x - \gamma_1 y), \quad y' = y(\alpha_2 - \beta_2 y - \gamma_2 x).$$

**Fact 9.4.2**

Four outcomes: (1)  $x$  wins, (2)  $y$  wins, (3) coexistence at a stable interior equilibrium, (4) competitive exclusion (winner depends on initial conditions).

## 9.5 Predator-Prey Equations

**Definition 9.5.1 (Lotka-Volterra Predator-Prey)**

$x = \text{prey}$ ,  $y = \text{predator}$ :

$$x' = \alpha x - \beta xy, \quad y' = -\gamma y + \delta xy \quad (\alpha, \beta, \gamma, \delta > 0)$$

**Fact 9.5.1 (Equilibria)**

$(0, 0)$  (extinction) and  $(\gamma/\delta, \alpha/\beta)$  (coexistence). The Jacobian is  $J = \begin{bmatrix} \alpha - \beta y & -\beta x \\ \delta y & -\gamma + \delta x \end{bmatrix}$ .

- $(0, 0)$ : eigenvalues  $\alpha, -\gamma$  (opposite sign)  $\Rightarrow$  **Saddle**.
- $(\gamma/\delta, \alpha/\beta)$ : eigenvalues  $\pm i\sqrt{\alpha\gamma} \Rightarrow$  **Center** for linearized. Nonlinear system has closed orbits (periodic population cycles).

**Example 9.5.2**

Analyze  $x' = 200x - 4xy$ ,  $y' = -150y + 2xy$ .

*Solution.*  $x(200 - 4y) = 0$  and  $y(-150 + 2x) = 0$  give equilibria  $(0, 0)$  and  $(75, 50)$ .

$$J = \begin{bmatrix} 200 - 4y & -4x \\ 2y & -150 + 2x \end{bmatrix}.$$

At  $(0, 0)$ :  $J = \begin{bmatrix} 200 & 0 \\ 0 & -150 \end{bmatrix}$ , eigenvalues  $200, -150 \Rightarrow$  **Saddle**.

At  $(75, 50)$ :  $J = \begin{bmatrix} 0 & -300 \\ 100 & 0 \end{bmatrix}$ ,  $\lambda^2 + 30000 = 0$ ,  $\lambda = \pm i\sqrt{30000} \Rightarrow$  **Center** (linearized). □

**Example 9.5.3**

Analyze  $x' = x(6 - 2x - 3y)$ ,  $y' = y(4 - x - 2y)$  (competing species).

*Solution.* Equilibria:  $(0, 0)$ ,  $(3, 0)$ ,  $(0, 2)$ , and the interior point from  $6 - 2x - 3y = 0$ ,  $4 - x - 2y = 0$ . Solving:  $x = 0$ ,  $y = 2$  — but that's already listed. Or from the linear system:  $2x + 3y = 6$  and  $x + 2y = 4$ , so  $x = 0$ ,  $y = 2$ . No interior coexistence point here.

$$J = \begin{bmatrix} 6 - 4x - 3y & -3x \\ -y & 4 - x - 4y \end{bmatrix}.$$

At  $(0, 0)$ : eigenvalues  $6, 4 \Rightarrow$  **Unstable node**.

At  $(3, 0)$ :  $J = \begin{bmatrix} -6 & -9 \\ 0 & 1 \end{bmatrix}$ , eigenvalues  $-6, 1 \Rightarrow$  **Saddle**.

At  $(0, 2)$ :  $J = \begin{bmatrix} 0 & 0 \\ -2 & -4 \end{bmatrix}$ , eigenvalues  $0, -4 \Rightarrow$  degenerate, needs further analysis. □